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Axially symmetric cosmological models for the perfect fluid

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Abstract. Families of metrics are found in synchronous coordinates generalising Tolman's spherically symmetric solutions, both for non-coherent dust and for special cases of perfect fluid with non-zero pressure. New metrics depend on an additional arbitrary function of R, θ, and they lead specifically to solutions of the Schwarzschild and Friedmann types (but differing from them in general, especially in their Petrov types and isometries).

1. Introduction

In relativistic cosmology, a number of isotropic and homogeneous models of the universe are studied, which often admit representation in some generalisations of spherical coordinates. These models, along with the other spherically symmetric solutions of the Einstein equations, form the simplest descriptions of observed material systems and the space–times thereof. However, observational cosmology tells us that even in the large scale picture of the universe, spherical symmetry may be merely an approximation. There are hints on the existence of some preferred polarisation of the angular momenta of galaxies, and even of linear structures of extremely large scale in the universe. Though the real structure of our world cannot be characterised exactly by some strict symmetry, it appears to be adequate to study more closely also axially symmetric cases which are richer in physical content, especially since in spherically symmetric systems there cannot exist gravitational radiation, dragging phenomena, etc. Some progress has already been made, on the one hand, in approximate study of systems only slightly declining from spherical symmetry; these were investigated using perturbation methods. On the other hand, the exact axial symmetry has been treated extensively for some localised sources. We consider here a generalisation of the latter case, but treat it from the cosmological point of view which does not exclude localised sources. We show here that using synchronous coordinates, as was done by Tolman (1934), Dandach and Mitskiévic (1980a, b) and Dandach (1982) for the spherical symmetry, one is able to solve Einstein's equations for axially symmetric cosmological models. (Such models as well as other non-stationary but not rotating axisymmetric non-vacuum metrics were not discussed by Kramer *et al* (1980), since they escaped general attention.) The axial symmetry is described by a space-like Killing vector field ξ with closed (compact) trajectories; this Killing vector vanishes on the axis of rotation. Hence, one may introduce coordinates such that the metric coefficients are independent of the coordinate φ ($\xi = \partial_\varphi$). However, we shall not use the coordinates of Weyl, Levi-Civitä, Papapetrou or others, but the synchronous coordinates that are more convenient for the description of cosmological models. The transition to synchronous coordinates in static or stationary metrics results generally in the off-diagonal spatial metric (see Gupta and Mitskiévic 1983), so the assumption of the diagonal metric (rotation is absent)

$$ds^2 = dT^2 - A^2(R, \theta, T)\, dR^2 - B^2(R, \theta, T)\, d\theta^2 - D^2(R, \theta, T)\, d\varphi^2, \tag{1}$$

that we use below, imposes supplementary limitation. We shall see, however, that this fact does not prevent us from getting physically reasonable solutions. Evidently, in (1), as in any synchronous system, time coordinate lines are geodesics.

2. The field equations

Let us introduce an orthonormal covector basis $\theta^{(\alpha)}$ with

$$ds^2 = \theta^{(0)}\theta^{(0)} - \theta^{(1)}\theta^{(1)} - \theta^{(2)}\theta^{(2)} - \theta^{(3)}\theta^{(3)},$$

so that

$$\theta^{(0)} = dT, \qquad \theta^{(1)} = A\, dR, \qquad \theta^{(2)} = B\, d\theta, \qquad \theta^{(3)} = D\, d\varphi. \tag{2}$$

Using the first and second Cartan equations of structure

$$d\theta^{(\alpha)} = -\omega^{(\alpha)}{}_{(\beta)} \wedge \theta^{(\beta)}, \qquad \Omega^{(\alpha)}{}_{(\beta)} = d\omega^{(\alpha)}{}_{(\beta)} + \omega^{(\alpha)}{}_{(\gamma)} \wedge \omega^{(\gamma)}{}_{(\beta)} \tag{3}$$

we get all the components of the curvature tensor $R^{(\mu)}{}_{(\nu)(\sigma)(\delta)}$, the Ricci tensor $R_{(\mu)(\nu)}$ and the scalar curvature R. This gives us the left-hand part of the Einstein equations,

$$R_{(\mu)(\nu)} - \tfrac{1}{2} g_{(\mu)(\nu)} R = -x T_{(\mu)(\nu)}; \tag{4}$$

the right-hand part we shall fix from physical considerations. We have then the system of equations

$$-xT_{(0)(1)} = \frac{1}{B}\left(\frac{B'}{A}\right)^{\cdot} + \frac{1}{D}\left(\frac{D'}{A}\right)^{\cdot}, \tag{5a}$$

$$-xT_{(1)(2)} = \frac{1}{BD}\left[\left(\frac{D'}{A}\right)^{\circ} - \frac{B'}{A}\frac{\mathring{D}}{B}\right], \tag{5b}$$

$$-xT_{(0)(2)} = \frac{1}{A}\left(\frac{\mathring{A}}{B}\right)^{\cdot} + \frac{1}{D}\left(\frac{\mathring{D}}{B}\right)^{\cdot}, \tag{5c}$$

$$-xT_{(1)(1)} = \frac{\ddot{B}}{B} + \frac{\ddot{D}}{D} - \frac{1}{BD}\left[\frac{B'}{A}\frac{D'}{A} + \left(\frac{\mathring{D}}{B}\right)^{\circ} - \dot{B}\dot{D}\right], \tag{5d}$$

$$-xT_{(2)(2)} = \frac{\ddot{A}}{A} + \frac{\ddot{D}}{D} - \frac{1}{AD}\left[\left(\frac{D'}{A}\right)' + \frac{\mathring{A}}{B}\frac{\mathring{D}}{B} - \dot{A}\dot{D}\right], \tag{5e}$$

$$-xT_{(3)(3)} = \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} - \frac{1}{AB}\left[\left(\frac{B'}{A}\right)' + \left(\frac{\mathring{A}}{B}\right)^{\circ} - \dot{A}\dot{B}\right], \tag{5f}$$

$$-xT_{(0)(0)} = \frac{1}{AB}\left[\left(\frac{B'}{A}\right)' + \left(\frac{\mathring{A}}{B}\right)^{\circ} - \dot{A}\dot{B}\right] + \frac{1}{BD}\left[\frac{B'}{A}\frac{D'}{A} + \left(\frac{\mathring{D}}{B}\right)^{\circ} - \dot{B}\dot{D}\right] + \frac{1}{AD}\left[\left(\frac{D'}{A}\right)' + \frac{\mathring{A}}{B}\frac{\mathring{D}}{B} - \dot{A}\dot{D}\right], \tag{5g}$$

where

$$(\;)^{\cdot}=\partial/\partial T, \qquad (\;)'=\partial/\partial R, \qquad (\;)^{\circ}=\partial/\partial\theta.$$

For the energy tensor we choose that of the perfect fluid,

$$T_{(\mu)(\nu)}=(\rho+p)u_{(\mu)}u_{(\nu)}-pg_{(\mu)(\nu)}, \tag{6}$$

with an as yet unspecified equation of state.

If in the fluid under consideration there exists a pressure which depends on time only, then it is always possible to choose the co-moving synchronous system of coordinates. In fact, substituting (6) into the conservation equation

$$T^{\mu\nu}_{;\nu}=0, \tag{7}$$

we get

$$\rho u^{\nu}u^{\mu}_{;\nu}+(\rho u^{\nu})_{;\nu}u^{\mu}+p_{;\nu}(u^{\mu}u^{\nu}-g^{\mu\nu})+p(u^{\mu}u^{\nu})_{;\nu}=0.$$

Because of $u_{\mu}u^{\mu}=1$ and $u_{\mu}u^{\mu}{}_{;\nu}=0$ we have then

$$(\rho+p)u^{\mu}{}_{;\nu}u^{\nu}+p_{;\nu}(u^{\mu}u^{\nu}-g^{\mu\nu})=0,$$

$p_{,\nu}(u^{\mu}u^{\nu}-g^{\mu\nu})$ being the pressure gradient projected onto the three-space. If the pressure does not depend on the spatial coordinates the latter expression vanishes, and we get the equation $(\rho+p)u^{\mu}{}_{;\nu}u^{\nu}=0$ describing the geodesic motion if $\rho+p\neq 0$. Hence, without loss of generality it is possible to choose a frame of reference such that the three-velocity of the matter will be zero everywhere ($u^{i}=0$, $i=1,2,3$ and $u^{0}=1$). The energy–momentum tensor has then a simple form

$$T_{(\mu)(\nu)}=(\rho+p)u_{(\mu)}u_{(\nu)}-pg_{(\mu)(\nu)}=\rho\delta^{0}_{\mu}\delta^{0}_{\nu}+p\delta^{j}_{i}. \tag{8}$$

3. A study of the equations

It follows from (8) that the off-diagonal expressions in the system (5) ($5(a)$–$(5c)$) must vanish, the right-hand sides of $(5d)$–$(5f)$ being equal to $(-xp)$, and the right-hand side of $(5g)$ equal to $(-x\rho)$. Then for $(5a)$, we get

$$(1/B)(B'/A)^{\cdot}+(1/D)(D'/A)^{\cdot}=0 \tag{5a'}$$

where the functions A, B, D are positive square roots of the moduli of the metric tensor components.

Equation $(5a')$ has one evident solution for $(B'/A)^{\cdot}=(D'/A)^{\cdot}$, namely,

$$B'/A=f_1(R,\theta), \qquad D'/A=f_2(R,\theta). \tag{9a}$$

Other solutions emerge in two cases:

$$(B'/A)^{\cdot}>0 \qquad \text{and} \qquad (D'/A)^{\cdot}<0, \tag{9b}$$

$$(B'/A)^{\cdot}<0 \qquad \text{and} \qquad (D'/A)^{\cdot}>0. \tag{9c}$$

One may assume in a special case

$$D=\varepsilon(R,\theta)B\Rightarrow 2B'/A=-(\varepsilon'/\varepsilon)(B/A)+K(R,\theta)$$

for

$$B'/A=U(R,\theta)S(T)\Rightarrow K=0\Rightarrow 2B'/B=-\varepsilon'/\varepsilon.$$

In this paper we consider generally the case ($9a$); case ($9b$) is studied for the homogeneous universe only, where

$$\dot{A}/A = \dot{B}/B = \dot{D}/D.$$

Then from (5.2) it immediately follows that

$$\left(\frac{D'}{A}\right)^{\circ} = \frac{B'}{A}\frac{D^{\circ}}{B}, \qquad \frac{\mathring{D}}{B} = \frac{\mathring{f}_2}{f_1} = \varphi(R, \theta), \qquad \frac{\mathring{A}}{B} = \frac{\varphi'}{f_2} = \psi(R, \theta). \tag{10}$$

The expression ($5c$) automatically vanishes as a result of ($9a$) and (10); ($5g$) we solve below separately, that gives the value of mass–energy density ρ of the system. The remaining equations ($5d$)–($5f$) are mutually connected in the following way:

$$(-xT_{(1)(1)}BD)' = -xT_{(2)(2)}B'D - xT_{(3)(3)}BD'.$$

When ($5e$), ($5f$) are integrated over R, the solution of ($5d$)–($5f$) can be found by solving two equations:

$$\begin{aligned} &2B\ddot{B} + \dot{B}^2 + xpB^2 = f_1^2 + 2\int f_1\psi^0 \,\mathrm{d}R + z_1(T, \theta), \\ &2D\ddot{D} + \dot{D}^2 + xpD^2 = f_2^2 + 2\int f_2\varphi\psi \,\mathrm{d}R + z_2(T, \theta). \end{aligned} \tag{11}$$

The metric (1) is axially symmetric, so it admits spherical symmetry as a particular case, transition to which consists of dropping the dependence of A, B on the angle θ, while $D \to B \sin\theta$. Such a transition should be possible at every step of our analysis (and it changes both the isometry group and the Petrov type of our metrics). In terms of the independent functions obtained earlier (f_1, f_2), from which all other functions (A, B, D, φ, ψ) are to be constructed, the transition to the spherically symmetric case means that

$$f_2 = f_1 \sin\theta \tag{12}$$

where f_1 depends on R only. It follows from (12) that

$$\varphi = \cos\theta, \qquad \psi = \frac{\varphi'}{f_2} = 0 = \frac{\mathring{A}}{B} \Rightarrow \mathring{A} = 0 = \frac{B'^{\circ}}{f_1} \tag{13}$$

whence $\mathring{B} = 0$.

As a result of (12), $z_1 = -1$, $z_2 = -\sin^2\theta$ (in the case of spherical symmetry); indeed, substituting (12) into ($5c$) and taking into account (10) and (13), we should get (see Ryabushko 1967)

$$2B\ddot{B} + \dot{B}^2 + xpB^2 = f_1^2 - 1, \qquad 2D\ddot{D} + \dot{D}^2 + xpD^2 = f_2^2 - \sin^2\theta,$$

whence

$$\begin{aligned} &f_1^2 + 2\int f_1\psi^{\circ}\,\mathrm{d}R + z_1 = f_1^2 - 1, && \psi^{\circ} = 0, \\ &f_2^2 + 2\int f_2\varphi\psi\,\mathrm{d}R + z_2 = f_2^2 - \sin^2\theta, && \psi = 0. \end{aligned} \tag{14}$$

From (12) and (14) it follows that the assumption of the independence of f_1 and f_2 of T leads to the independence of z_1 and z_2 of T. This holds for any f_1 and f_2

satisfying condition (12), so it may be assumed that z_1 and z_2 do not depend in general on T. Then we rewrite the system (11) in the form

$$2X_n\ddot{X}_n+\dot{X}_n^2+xpX_n^2=F_n \tag{15}$$

where

$$n=1,2, \qquad X_1=B, \qquad X_2=D,$$

$$F_1=f_1^2+2\int f_1\psi^\circ\,\mathrm{d}R+z_1(\theta), \qquad F_2=f_2^2+2\int f_2\varphi\psi\,\mathrm{d}R+z_2(\theta). \tag{16}$$

4. Some types of solutions of (15)

Let us consider solutions of (15) in three cases.

(I) $p=0$

Integrating (15) over T once, we get

$$\dot{X}_n^2=M_n(R,\theta)/X_n+F_n(R,\theta) \tag{16a}$$

where $M_n(R,\theta)$ are arbitrary integration functions.

Equations (16a) have three classes of solutions:

$$\text{(a)}\quad F_n=0;\ X_n=[\tfrac{3}{2}(M_n)^{1/2}(\phi_n\pm T)]^{2/3}; \tag{17}$$

$$\text{(b}_1\text{)}\quad F_n>0;\ X_n=(M_n/2F_n)(\cosh\eta_n-1),\ (M_n/2F_n^{3/2})(\sinh\eta_n-\eta_n)=\phi_n\pm T; \tag{18}$$

$$\text{(b}_2\text{)}\quad F_n<0;\ X_n=[M_n/2(-F_n)](1-\cos\eta_n),\ [M_n/2(-F_n)^{3/2}](\eta_n-\sin\eta_n)=\phi_n\pm T; \tag{19}$$

where $\phi_n(R,\theta)$ are new integration functions (independent of T).

(II) $p=\text{constant}\neq 0$

Integrating (15) over T once, we have

$$\dot{X}_n^2=M_n/X_n-\tfrac{1}{3}xpX_n^2+F_n. \tag{20}$$

It is possible to integrate (20) for $F_n=0$ and, in special cases, for:

(a) $F_n=0$;

$$X_n=(3M_n/xp)^{1/3}\sin^{2/3}[\tfrac{3}{2}(\tfrac{1}{3}xp)^{1/2}(\phi_n\pm T)], \qquad M_n>0,\ p>0, \tag{21}$$

$$X_n=(-3M_n/xp)^{1/3}\sinh^{2/3}[\tfrac{3}{2}(-\tfrac{1}{3}xp)^{1/2}(\phi_n\pm T)], \qquad M_n>0,\ p<0, \tag{22}$$

$$X_n=(3M_n/xp)^{1/3}\cosh^{2/3}[\tfrac{3}{2}(-\tfrac{1}{3}xp)^{1/2}(\phi_n\pm T)], \qquad M_n<0,\ p<0. \tag{23}$$

For $p\to 0$, (21) and (22) turn asymptotically into (17).

(b) $F_n=(\tfrac{9}{4}xpM_n)^{1/3}$;

$$\pm(\phi_n+T)=\left(\frac{4}{xp}\right)^{1/2}\left[\sqrt{3}\sin^{-1}\left(\frac{X_n}{2\beta}\right)^{1/2}-\sin^{-1}\left(\frac{3}{2}\frac{X_n}{\beta+X_n}\right)^{1/2}\right] \tag{24}$$

where $\beta=(3M_n/2xp)^{1/3}$.

(III) $p=p(T)$

$$\text{(a)}\quad F_n=0;\ X_n=X_0\exp\int(\tfrac{1}{3}xp)^{1/2}\cot[\tfrac{3}{2}(\tfrac{1}{3}xp)^{1/2}(\phi+T)]\,\mathrm{d}T. \tag{25}$$

(a$_1$) $p = a/T^2, \qquad a = \text{constant} > 0;$

$$X_n = X_0|T|^{1/3}\sin^{2/3}[\tfrac{1}{2}(3xa-1)^{1/2}\ln(\phi/T)], \qquad 3xa-1>0, \tag{26}$$

$$X_n = X_0|T|^{1/3}\sinh^{2/3}[\tfrac{1}{2}(1-3xa)^{1/2}\ln(\phi/T)], \qquad 3xa-1<0, \tag{27}$$

$$X_n = X_0|T|^{1/3}\ln^{2/3}(\phi/T), \qquad 3xa-1=0. \tag{28}$$

(a$_2$) $p = a/T^{4/3}, \qquad a = \text{constant} > 0;$

$$X_n = X_0[\cos(\gamma+\phi)+\gamma\sin(\gamma+\phi)]^{2/3}, \tag{29}$$

$$\gamma = \tfrac{3}{2}(3xa)^{1/2}T^{1/3}.$$

(a$_3$) $p = a/(T^2+1)^2, \qquad a = \text{constant} > 0;$

$$X_n = X_0(T^2+1)^{1/3}\sin^{2/3}[\phi+(1+\tfrac{3}{4}xa)^{1/2}\cot^{-1}T]. \tag{30}$$

For case (IIIb) ($F_n \neq 0$ and $p = p(T)$) equations (15) are not integrable in their general form when no special dependence of p on T and the special form of F_n are fixed. From (5g) and (8) we get in the general case an expression for the mass–energy density

$$\begin{aligned}\rho &= \frac{1}{x}\left[\frac{\dot{B}'}{B'}\left(\frac{\dot{B}}{B}+\frac{\dot{D}}{D}\right)+\frac{\dot{B}\dot{D}}{BD}-\frac{f_1f_1'+f_1(\varphi'/f_2)^\circ}{BB'}-\frac{f_2f_2'+\varphi\varphi'}{DD'}-\frac{f_1f_2+\mathring{\varphi}}{BD}\right]\\ &= -3p-(2/x)(\ddot{A}/A+\ddot{B}/B+\ddot{D}/D).\end{aligned} \tag{31}$$

If the condition (12) is satisfied, and we pass to the case of spherical symmetry, then for dust ($p=0$) we have the Tolman case, and from (17)–(19) we obtain the three well known Friedmann solutions for the universe with negative, zero and positive curvature of the three-space, as well as the Schwarzschild solution in the Lemaître coordinates. The condition (12) with $p = p(T)$ leads to the case of spherical symmetry with non-zero pressure (Dandach 1982, Ryabushko 1967). Substituting into (31) equations (16) and (20), we get for the mass–energy density ρ correspondingly:

$$p=0, \qquad \rho = (1/2x)(M_1'/B'B^2+M_2'/D'D^2), \tag{32}$$

$$p = \text{constant}, \qquad \rho = \frac{1}{2x}\left(\frac{M_1'}{BB'}+\frac{M_2'}{DD'}\right)-p, \tag{33}$$

$$p = p(T),$$

$$\begin{aligned}x\rho = {}&\dot{N}_1\frac{B_0'\dot{N}_1+B_0N_1'\dot{N}_1}{B_0'+B_0N_1'}+\dot{N}_2\frac{D_0'\dot{N}_2+D_0\dot{N}_2'+D_0N_2'\dot{N}_2}{D_0'+D_0N_2'}\\ &-\tfrac{1}{2}\dot{N}_2'/D_0^2e^{2N_2}-2\ddot{N}_1-2\dot{N}_1^2-2\ddot{N}_2-2\dot{N}_2^2-2xp,\end{aligned} \tag{34}$$

where

$$N_2 = \int(\tfrac{1}{3}xp)^{1/2}\cot[\tfrac{3}{2}(\tfrac{1}{3}xp)^{1/2}(T+\phi_n)]\,\mathrm{d}T, \qquad X_n = X_0\exp N_2.$$

5. Analogues of the Schwarzschild and Friedmann fields

To construct the axially symmetric analogue of the Schwarzschild solution, one has to put $\rho=0$ for $p=0$; then from (32), $M_1'=M_2'=0$. Substituting the latter equality into (17), we get

$$B=[\tfrac{3}{2}(M_1(\theta))^{1/2}(\phi_1(R,\theta)\pm T)]^{2/3}, \qquad D=[\tfrac{3}{2}(M_2(\theta))^{1/2}(\phi_2(R,\theta)\pm T)]^{2/3},$$

$$A=\frac{B'}{f_1}=\frac{1}{f_1}\sqrt{M_1}\phi_1'[\tfrac{3}{2}\sqrt{M_1}(\phi_1\pm T)]^{-1/3}. \tag{35}$$

In the Friedmann-like case we deal with a homogeneous universe. The observable local expansion velocity depends on time only (Peebles 1971),

$$H_R=\frac{\dot{A}}{2A}=g_A(T), \qquad H_\theta=\frac{\dot{B}}{2B}=g_B(T), \qquad H_\varphi=\frac{\dot{D}}{2D}=g_D(T), \tag{36}$$

where H is the direction-dependent Hubble function. Taking into account (9), we get from (36) analogues of all three Friedmann models with zero, negative and positive three-curvature: for zero three-curvature

$$\mathrm{d}s^2=\mathrm{d}T^2-a^2(T)[(M_1^{-4/3}M_1'^2/9f_1)\,\mathrm{d}R^2+M_1^{2/3}\,\mathrm{d}\theta^2+M_2^{2/3}\,\mathrm{d}\varphi^2] \tag{37}$$

with M_1 and M_2 such that $(M_1^{1/3})'/(M_2^{1/3})'=f_2/f_1$ and for non-zero three-curvature

$$\mathrm{d}s^2=\mathrm{d}T^2-a^2(T)\left[\left(\frac{M_1}{2F_1}\right)'^2\frac{1}{f_1^2}\mathrm{d}R^2+\left(\frac{M_1}{2F_1}\right)^2\mathrm{d}\theta^2+\left(\frac{M_2}{2F_2}\right)^2\mathrm{d}\varphi^2\right] \tag{38}$$

(these solutions are now not conformally flat: they are of the algebraically general Petrov type I). From (17) we get the Schwarzschild solution in Lemaître coordinates if

$$M_n=2m, \qquad \phi_n=R;$$

when $\phi_n=$ constant, the homogeneous case takes place. Vacuum solutions, relevant to (17), correspond to the case when $M_n'=0$. Then in view of (32), the mass–energy density vanishes ($\rho=0$). Choosing

$$\phi_n=RE^\gamma(R), \qquad M_1=(2m)^{1/2}C^\alpha(\theta), \qquad M_2=(2m)^{1/2}L^\beta(\theta)C^\alpha(\theta),$$

we get such an inhomogeneous model, generalising the Schwarzschild case (a 'prolate' singularity), that reduces to the Schwarzschild solution, if $\alpha=\beta=\gamma=0$. In this case

$$\mathrm{d}s^2=\mathrm{d}T^2-A^2(R,\theta,T)\,\mathrm{d}R^2-B^2(R,\theta,T)\,\mathrm{d}\theta^2-D^2(R,\theta,T)\,\mathrm{d}\varphi^2$$

where

$$B=[\tfrac{3}{2}(2m)^{1/2}C^\alpha(RE^\gamma\mp T)]^{2/3}, \qquad D=[\tfrac{3}{2}(2m)^{1/2}L^\beta C^\alpha(RE^\gamma\mp T)]^{2/3},$$

$$A=B'/f_1. \tag{39}$$

The expressions (35) and (39) have to satisfy the conditions (10), in particular

$$D'/B'=f_2/f_1(R,\theta), \qquad \mathring{A}/B=\psi(R,\theta).$$

The first of these conditions results in $C'=0$; from the second condition and from (39) we get

$$\psi=\frac{2}{9}\frac{E^\gamma+\gamma E^{\gamma-1}E'R}{E^\gamma R-T}\left[3\left(\frac{1}{f_1}\right)^\circ+2\alpha\frac{\mathring{C}}{C}\right],$$

then $f_1=K(R)C^{3/2\alpha}$, hence $\mathring{A}=0$.

It is easily seen that the corresponding space–time is of Petrov type I, and it degenerates to type D only in the case of strict spherical symmetry (12). As to the

Killing vectors, their components ξ^0 and ξ^1 can be non-zero only if $\partial f_1/\partial R = 0$ (in this case $\phi_1 = \phi_2$, $\phi' = \pm 1$ in (35)), and then $\xi^0 = a = \text{constant}$, $\xi^1 = -a/\phi'$; otherwise $a \equiv 0$, and the solutions cannot be static or stationary. On the other hand,

$$\xi^2 = hC^{-2\alpha/3} \sin[\sqrt{b}(\varphi + \varphi_0)]$$

and

$$\xi^3 = K - h\sqrt{b} \cos[\sqrt{b}(\varphi + \varphi_0)] \int C^{-2\alpha/3} \sin^{4\beta/3}\theta \, d\theta,$$

where b, K, $h = \text{constant}$, $l = \sin[\sqrt{b}(\varphi + \varphi_0)]$, and the function C, which comes here from (35), has to satisfy the equation

$$d^2z/d\theta^2 - \tfrac{2}{3}\beta \cot\theta \, dz/d\theta + (2\beta/3 \sin\theta - b \sin^{-4\beta/3}\theta)z = 0, \tag{40}$$

$z = C^{-2\alpha/3}$; otherwise $h = 0$, and we remain with $\bar{\xi} = \partial_\varphi$ $(K = 1)$ which exists in the general axisymmetric case when no other Killing vector is present. This analysis shows that the solutions discussed here are more general than those which can be found in Kramer *et al* (1980). If, however, (40) holds, we come to non-trivial extra Killing vectors of the form $\bar{\xi} = \xi^2 \partial_\theta + \xi^3 \partial_\varphi$ with two independent choices of the phase φ_0 leading to $l = \sin\varphi$ and $l = \cos\varphi$ (which usually represent the spherical symmetry). Returning to the case $\partial f_1/\partial R = 0$ (i.e. $a \neq 0$) we can choose $a = 1$, and since $\phi' = 1$, we get $\xi = \partial_T \mp \partial_R$. Hence

$$\bar{\xi} \cdot \bar{\xi} = g_{TT} + g_{RR} = 1 - A^2 \gtrless 0.$$

In the region where the upper non-equality holds, the solution is static (or stationary if the Killing congruence rotates); where the lower non-equality holds, a non-stationary region is realised, and these two regions are divided one from another by the usual horizon (at the hypersurface $\bar{\xi} \cdot \bar{\xi} = 0$), but this cannot be the general case for the solution (35).

6. Conclusions

The analysis presented above has shown that in the axially symmetric case in synchronous coordinates there exists a situation similar to Tolman's well known spherically symmetric case (but to some extent richer than it was). We have found families of metrics (see (17)–(19), (21)–(23), (25)–(30), (32)–(34)) which generalise spherically symmetric ones (both for non-coherent dust and for special cases of perfect fluid with non-zero pressure); they depend on one extra (as compared with the spherically symmetric fields) arbitrary function of R and θ. Specifically, our metrics lead to solutions of the Schwarzschild ((35), (39)) and Friedmann ((37), (38)) types but differing from them in general. A limiting transition (see (12) and the corresponding discussion) is studied which leads from the axially symmetric exact solutions to the spherically symmetric ones.

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